A GENERALIZATION OF JUNG'S THEOREM

M. Henk

Abstract. The theorem of Jung establishes a relation between circumradius and diameter of a convex body. The half of the diameter can be interpreted as the maximum of circumradii of all 1-dimensional sections or 1-dimensional orthogonal projections of a convex body. This point of view leads to two series of j-dimensional circumradii, defined via sections or projections. In this paper we study some relations between these circumradii and by this we find a natural generalization of Jung's theorem.

Introduction

Throughout this paper E^d denotes the d-dimensional euclidean space and the set of all convex bodies $K \subset E^d$ — compact convex sets — is denoted by $\mathcal{K}^d$. The affine (convex) hull of a subset $P \subset E^d$ is denoted by $\mathrm{aff}(P)$ ($\mathrm{conv}(P)$) and $\dim(P)$ denotes the dimension of the affine hull of P. The interior of P is denoted by $\mathrm{int}(P)$ and $\mathrm{relint}(P)$ denotes the interior with respect to the affine hull of P. $\|\cdot\|$ denotes the euclidean norm and the set of all i-dimensional linear subspaces of E^d is denoted by $\mathcal{L}_i^d$. $L^\perp$ denotes for $L \in \mathcal{L}_i^d$ the orthogonal complement and for $K \in \mathcal{K}^d$, $L \in \mathcal{L}_i^d$ the orthogonal projection of K onto L is denoted by $K|L$.

The diameter, circumradius and inradius of a convex body $K \in \mathcal{K}^d$ is denoted by $D(K)$, $R(K)$ and $r(K)$, respectively. For a detailed description of these functionals we refer to the book [BF]. With this notation we can define the following i-dimensional circumradii

Definition. For $K \in \mathcal{K}^d$ and $1 \le i \le d$ let

$$\begin{aligned} &i) \quad R_\sigma^i(K) := \max_{L \in \mathcal{L}_i^d} \max_{x \in L^\perp} R(K \cap (x + L)), \\ &ii) \quad R_\pi^i(K) := \max_{L \in \mathcal{L}_i^d} R(K|L). \end{aligned}$$

We obviously have $R_\sigma^{i+1}(K) \ge R_\sigma^i(K)$, $R_\pi^{i+1}(K) \ge R_\pi^i(K)$, $R_\pi^i(K) \ge R_\sigma^i(K)$ and $R_\sigma^d(K) = R_\pi^d(K) = R(K)$, $R_\sigma^1(K) = R_\pi^1(K) = D(K)/2$.

The theorem of Jung [J] states a relation between the circumradius and the diameter of a convex body. On account of the definition of $R_\sigma^d(K)$, $R_\sigma^1(K)$ we can describe his result as follows

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Theorem of Jung. *Let $K \in \mathcal{K}^d$. Then*

$$R_\sigma^d(K) \le \sqrt{\frac{2d}{d+1}}\,R_\sigma^1(K), \tag{1.1}$$

and equality holds if and only if K contains a regular d-simplex with edge length $D(K)$.

In the same way the theorem may be described with the circumradii $R_\rho^d(K)$ and $R_\rho^1(K)$. Here we study in general the relations between the i-dimensional and j-dimensional circumradius of these both series and get the following results

Results

Theorem 1. *Let $K \in \mathcal{K}^d$ and $1 \le j \le i \le d$. Then*

$$R_\sigma^i(K) \le \sqrt{\frac{i(j+1)}{j(i+1)}}\,R_\sigma^j(K), \tag{1.2}$$

and equality holds for $i > j$ if and only if K contains a regular i-simplex with edge length $R_\sigma^j(K)\sqrt{\frac{2(j+1)}{j}}$.

Theorem 2. *Let $K \in \mathcal{K}^d$ and $1 \le j \le i \le d$. Then*

$$R_\pi^i(K) \le \sqrt{\frac{i(j+1)}{j(i+1)}}\,R_\pi^j(K), \tag{1.3}$$

and equality holds for $i > j$ if and only if an orthogonal projection of K onto an i-dimensional linear subspace contains a regular i-simplex with edge length $R_\pi^j(K)\sqrt{\frac{2(j+1)}{j}}$.

Let us remark that both theorems are a generalization of the classical theorem of Jung since for $i = d$, $j = 1$ the inequalities (1.2) and (1.3) become (1.1).

Proofs

To prove these theorems it is necessary to examine in more detail the circumradii of simplices since the circumradius of a convex body K is determined by the circumradius of a certain simplex $\overline{T} \subset K$. This well known fact is described in the following lemma

Lemma 1. *Let $K \subset \mathcal{K}^d$ and 0 be the center of the circumball of K. Then there exists a k-simplex $\overline{T} \subset K$, $\overline{T} = \mathrm{conv}(\{x^0, \dots, x^k\})$ with*

$$0 \in \mathrm{relint}(\overline{T}),\ R(\overline{T}) = R(K) \text{ and } \|x^i\| = R(K),\ 0 \le i \le k.$$

Proof. cf. [BF], p. 9 and p. 54. q.e.d.

With this lemma it is easy to find such $(d-1)$-dimensional planes for a simplex which produce the maximal $(d-1)$-circumradius with respect to projections or sections.

Lemma 2. *Let $T \in \mathcal{K}^d$ be a d-simplex, $\hat{F}$ a face of T with maximal circumradius and $\hat{L} \in \mathcal{L}^d_{d-1}$, $\hat{x} \in \hat{L}^\perp$ with $\hat{x} + \hat{L} = \operatorname{aff}(\hat{F})$. Then*

$$\begin{aligned} i) \quad & R^{d-1}_\sigma(T) = R(T \cap (\hat{x} + \hat{L})) = R(\hat{F}), \\ ii) \quad & R^{d-1}_\pi(T) = R(T|\hat{L}) = R(\hat{F}). \end{aligned}$$

Proof. Let $L_{d-1} \in \mathcal{L}^d_{d-1}$ with $R^{d-1}_\pi(T) = R(T|L_{d-1})$ and let $T|L_{d-1}$ the convex hull of the points $x^0, \dots, x^d$, where $x^0, \dots, x^d$ denote the images of the vertices of T under the projection onto L_{d-1}. Further let 0 be the center of the circumball of $T|L_{d-1}$ and $\overline{T} \subset T|L_{d-1}$, $\overline{T} = \operatorname{conv}(\{x^0, \dots, x^k\})$, $1 \le k \le d-1$, a k-simplex with the properties of lemma 1.

Now let F be a face of T containing such $k+1$ vertices which are mapped onto $x^0, \dots, x^k$ with respect to the orthogonal projection onto L_{d-1}. We have

$$R(\hat{F}) \ge R(F) \ge R(F|L_{d-1}) \ge R(\overline{T}) = R^{d-1}_\pi(T);$$

otherwise $R(\hat{F}) \le R^{d-1}_\sigma(T) \le R^{d-1}_\pi(T)$ and the assertion follows. q.e.d.

On account of the lemma above we have $R(S)/R^{d-1}_\pi(S) = R(S)/R^{d-1}_\sigma(S) = d/(d^2 - 1)^{1/2}$ for a regular d-simplex S. That this is even an upper bound for every simplex is shown in the next lemma.

Lemma 3. *Let $T \in \mathcal{K}^d$ a simplex. Then*

$$\begin{aligned} i) \quad & R(T) \le \frac{d}{\sqrt{d^2-1}} R^{d-1}_\sigma(T), \\ ii) \quad & R(T) \le \frac{d}{\sqrt{d^2-1}} R^{d-1}_\pi(T), \end{aligned}$$

and equality holds if and only if T is a regular d-simplex.

Proof. If T is a regular d-simplex we have equality by lemma 2. Hence on account of $R^{d-1}_\sigma(T) \le R^{d-1}_\pi(T)$ it suffices to prove the lemma for the $(d-1)$-circumradius $R^{d-1}_\sigma(T)$.

Let 0 be the center of the circumball of T and $\{x^0, \dots, x^k\}$ a suitable subset of the vertices of T, such that $\overline{T} = \operatorname{conv}(\{x^0, \dots, x^k\})$ has the properties of lemma 1. If $k < d$ then

$$R(T) = R(\overline{T}) = R^{d-1}_\sigma(T) < \frac{d}{\sqrt{d^2-1}} R^{d-1}_\sigma(T). \tag{2.1}$$

Hence we may assume that $T = \operatorname{conv}(\{x^0, \dots, x^d\})$ is a d-simplex with $0 \in \operatorname{int}(T)$ and $\|x^i\| = R(T)$, $0 \le i \le d$.

Let λ be the maximal radius of a d-dimensional ball with center 0, which is contained in T. This ball touches a face F of T in a point λa, $\|a\| = 1$. Let F be given by $\operatorname{conv}(\{x^1, \dots, x^d\})$. Since a is a normal vector of $\operatorname{aff}(F)$ we have

$$\|x^i - \lambda a\|^2 = R(T)^2 - \lambda^2, \quad 1 \le i \le d.$$

Hence λa is the center of the circumball of F [BF, p. 54] and it follows

$$R(T)^2 - R^{d-1}_\sigma(T)^2 \le \lambda^2. \tag{2.2}$$

For the inradius $r(T)$ of a simplex T we have $r(T) \leq R(T)/d$ [F] and so by the choice of λ

$$\lambda^2 \leq \frac{R(T)^2}{d^2}. \tag{2.3}$$

Along with (2.2) this shows the inequality i). If we have equality in the relation i) then from (2.1), (2.2) and (2.3) follows that T is a d-simplex with $r(T) = R(T)/d$. This is only possible if T is regular [F]. q.e.d.

Now we are able to prove the theorems.

Proof of **Theorem 1**. It obviously suffices to show the inequalities

$$R_\sigma^i(K) \leq \frac{i}{\sqrt{i^2-1}} R_\sigma^{i-1}(K), \quad 1 < i \leq d. \tag{2.4}$$

Since the circumradii are invariant with respect to translations we may assume that there is an i-dimensional linear subspace $L_i \in \mathcal{L}_i^d$ with $R_\sigma^i(K) = R(K \cap L_i)$ and 0 is center of the circumball of $K \cap L_i$. Moreover let $T \subset (K \cap L_i)$ a k-simplex with the properties of lemma 1. Denoting by $R_\sigma^{i-1}(T; L_i)$ the $(i-1)$-circumradius of T with respect to the euclidean space L_i we get from lemma 3

$$R(T) \leq \frac{i}{\sqrt{i^2-1}} R_\sigma^{i-1}(T; L_i). \tag{2.5}$$

By the choice of T we have $R(T) = R_\sigma^i(K)$ and since $R_\sigma^{i-1}(K) \geq R_\sigma^{i-1}(T; L_i)$ the inequalities (1.1) are shown.

If an inequality of (1.1) is satisfied with equality for $i > j$ we must have equality in (2.4) and (2.5). By lemma 3 this means that T is a regular i-simplex which satisfies the relation

$$R(T) = R_\sigma^i(K) = \sqrt{\frac{i(j+1)}{j(i+1)}} R_\sigma^j(K). \tag{2.6}$$

Since T is regular we have $R(T) = (i/(2i+2))^{1/2} D(T)$ and by (2.6) we see that T has the diameter (edge length) $R_\sigma^j(K)((2j+2)/j)^{1/2}$.

Now let T be a regular i-simplex contained in K with the given edge length. On account of (1.1) we get

$$R(T) = \sqrt{\frac{i}{2i+2}} D(T) = \sqrt{\frac{i(j+1)}{j(i+1)}} R_\sigma^j(K) \geq R_\sigma^i(K). \tag{2.7}$$

Clearly $R(T) \leq R_\sigma^i(K)$ and so we can replace '$\leq$' by '$=$' in (2.7). q.e.d.

Proof of **Theorem 2**. On account of lemma 3 the proof can be done in the same way as the proof of theorem 1. q.e.d.

Remarks

(1) If we replace the first maximum condition by a minimum condition in the definition of the circumradii we get two other series of i-circumradii which now start with the half of the width of a convex body. If we further replace the circumradius by the inradius we totally get four series of circumradii and four series of inradii. Some of these functionals are studied in *Computational Geometry* [GK]. For a survey of these generalized circumradii and inradii we refer to [H].

(2) Theorems involving inradius, circumradius, diameter and width have a long tradition in the geometry of convex bodies. In this context we refer to [BL], [BF], [E], [DGK].

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Mathematisches Institut, Universität Siegen, Hölderlinstrasse 3, D-W-5900 Siegen, Federal Republic of Germany.